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Algorithms and Probability **Session Sheet S4** **FS 2026**

For the exercise sessions on 12 March 2026.

Exercise S4.1 – *Perfect Bipartite Matching*

Let $G = (A\dot{\cup}B, E)$ be a bipartite k-regular graph (i.e. every vertex has degree exactly k).

(a) Show that G contains a perfect matching.

(b) Show that G contains k pairwise disjoint perfect matchings.
Hint: use (a)

Solution S4.1 – *Perfect Bipartite Matching*

(a) Because G is k regular, both A and B contain exactly $\frac{|E|}{k}$ vertices. Hence, it suffices to show that G contains a matching that covers A. We do this by checking Hall's condition. Let $X \subseteq A$. There are $k \cdot |X|$ edge incident to X. Each edge that is incident to X is also incident to the neighborhood $N(X)$ of X. There are $k \cdot |N(X)|$ edges incident to the neighborhood of X. Hence, $k \cdot |X| \leq k \cdot |N(X)|$, which implies $|X| \leq |N(X)|$. Thus, Hall's condition is fulfilled and G contains a matching that covers A, which is a perfect matching (because $|A| = |B|$).

(b) We show the statement by induction.
Base Case: If $k = 0$ the statement trivially holds because any graph without edges contains 0 pairwise disjoint perfect matchings.
Induction Hypothesis: Assume the statement holds for some fixed k (i.e., assume that for this k every k-regular bipartite graph contains k pairwise disjoint perfect matchings).
Induction Step: Consider a $k + 1$-regular bipartite graph $G = (V, E)$. By (a) this graph has a perfect matching M. The graph $G' = (V, E \setminus M)$ is k-regular (because removing M reduces every degree by exactly one). Hence, by our induction hypothesis, G' contains k pairwise disjoint perfect matchings. Together with M this makes $k + 1$ pairwise disjoint perfect matchings in G. Thus, the statement follows by induction.