ALGORITHMEN & DATENSTRUKTUREN - ÜBUNGSSTUNDE 3

Exercise 2.2 *Fibonacci numbers* **(1 point).**

There are a lot of neat properties of the Fibonacci numbers that can be proved by induction. Recall that the Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$ and the recursion relation $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. For example, $f_2 = 1$, $f_5 = 5$, $f_{10} = 55$, $f_{15} = 610$.

Prove that $f_n \geq \frac{1}{3} \cdot 1.5^n$ for $n \geq 1$.

In your solution, you should address the base case, the induction hypothesis and the induction step.

How to see that we need more than 1 BC?

The recursion for f_{n+1} uses both, f_n and f_{n-1} which is an indicator for the need of 2 BC.

We go quickly through the tasks proof:

BC: $n=1$ $f_n = 1 \geq \frac{1}{3} \cdot 1.5^1 = 0.5$ ✓

$n=2$ $f_n = 1 \geq \frac{1}{3} \cdot 1.5^2 = 0.75$ ✓

IH: Assume the property holds for some positive integer k and $k+1$, that is $f_k \geq \frac{1}{3} \cdot 1.5^k$ and $f_{k+1} \geq \frac{1}{3} \cdot 1.5^{k+1}$

IS: We now show it also holds for $k+2$

$$
\begin{aligned}
f_{k+2} &\overset{def}{=} f_{k+1} + f_k \\
&\overset{IH}{\geq} \frac{1}{3} \cdot 1.5^{k+1} + \frac{1}{3} \cdot 1.5^k \\
&= \frac{1}{3} \cdot 1.5^k (1.5 + 1) \\
&= \frac{1}{3} \cdot 1.5^k \cdot 2.5 \\
&\geq \frac{1}{3} \cdot 1.5^k \cdot 2.25 \\
&= \frac{1}{3} \cdot 1.5^k \cdot 1.5^2 \\
&= \frac{1}{3} \cdot 1.5^{k+2}
\end{aligned}
$$

Thus, the property is proven for every positive integer n by the principle of mathematical induction. □

Exercise 2.3

(b) This question does not count toward the bonus point. Find f and g as in Theorem 1 such that $f \leq O(g)$, but the limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. This proves that the first point of Theorem 1 provides a sufficient, but not a necessary condition for $f \leq O(g)$.

$f(n) = 2 + (-1)^n$ $g(n) = 1$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ doesn't exist. But $f(n) \leq 3g(n)$ $\forall n \Rightarrow f(n) \in O(g(n))$

Exercise 2.4 *Asymptotic growth of* $\sum_{i=1}^{n}\frac{1}{i}$ **(1 point).**

The goal of this exercise is to show that the sum $\sum_{i=1}^{n}\frac{1}{i}$ behaves, up to constant factors, as $\log(n)$ when n is large. Formally, we will show $\sum_{i=1}^{n}\frac{1}{i} \le O(\log n)$ and $\log n \le O(\sum_{i=1}^{n}\frac{1}{i})$ as functions from $\mathbb{N}_{\ge 2}$ to $\mathbb{R}^+$.

For parts (a) to (c) we assume that $n = 2^k$ is a power of 2 for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will generalise the result to arbitrary $n \in \mathbb{N}$ in part (d). For $j \in \mathbb{N}$, define

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i}.$$

(a) For any $j \in \mathbb{N}$, prove that $S_j \le 1$.

Hint: Find a common upper bound for all terms in the sum and count the number of terms.

$\frac{1}{i} \le \frac{1}{2^{j-1}} \quad \forall i \in \{2^{j-1}+1, \dots, 2^j\}$.

There are $2^j - (2^{j-1}+1) = 2^{j-1}$ summands.

$\Rightarrow \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} \le \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^{j-1}} = 2^{j-1} \cdot \frac{1}{2^{j-1}} = 1$

(b) Show that $n \ln n \le O(\ln(n!))$.

Hint: You can use the fact that $\left(\frac{n}{2}\right)^{\frac{n}{2}} \le n!$ *for* $n \in \mathbb{N}_{\ge 2}$ *without proof.*

Monotonität von ln: $\ln(n!) \ge \ln\left(\frac{n}{2}^{\frac{n}{2}}\right) = \frac{n}{2}(\ln(n) - \ln(2))$

$2\ln(n!) + n\ln(2) \ge n \cdot \ln(n)$

Dann ausserdem: $n\ln(2) \le \ln 2 + \sum_{i=2}^{n} \ln(i) = \ln(2) + \ln(n!) \le 2\ln(n!)$

Demnach $n\ln(n) \le 2\ln(n!) + n\ln(2) \le 4\ln(n!)$

Prüfungsaufgaben

Algorithm 1

for $j = 1, \ldots, n$ **do**
  for $k = j, \ldots, n$ **do**
    $f()$

$$\sum_{j=1}^{n} \sum_{k=j}^{n} 1 = \sum_{j=1}^{n} j = \frac{n(n+1)}{2} = \Theta(n^2)$$

Algorithm 2

for $j = 1, \ldots, n$ **do**
  for $k = 1, 2, 3, 4, 5, \ldots, 2^j$ **do**
    for $l = 1, \ldots, 42$ **do**
      $f()$

$$\sum_{j=1}^{n} \sum_{k=1}^{2^j} \sum_{l=1}^{42} 1 = \sum_{j=1}^{n} \sum_{k=1}^{2^j} 42 = 42 \sum_{j=1}^{n} 2^j = 42 \cdot (2^{n+1} - 2) = \Theta(2^n)$$

	Claim	true	false	
1.	$5n^{2.5} + 2n^2 + n \leq \mathcal{O}(n^3)$	☒	☐	n^3 dominiert über $n^{2.5}$
2.	$\sqrt{n} \geq \Omega(\frac{n}{\log n})$	☐	☒	$\lim_{n\to\infty} \frac{n/\log n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$
3.	$\log_4 n^4 = \Theta(\log_6 n^6)$	☒	☐	$4 \log n = 6 \log n$
4.	$\sum_{i=1}^{n} \log_2 i = \Theta(n \log n)$	☒	☐	$\sum_{i=1}^{n} \log_2 i \leq O(n \log(n))$ & *
5.	$\sum_{i=1}^{n} \sqrt{i} \cdot \log_2^2 i \geq \Omega(n\sqrt{n} \log^2 n)$	☒	☐	$\sum_{i=1}^{n} \sqrt{i} \log_2^2 i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \sqrt{i} \log_2^2 i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \sqrt{\frac{n}{2}} \log_2^2 \frac{n}{2} \geq \Omega(n\sqrt{n} \log^2 n)$

n einsetzen bei Summe

$$* \sum_{i=1}^{n} \log_2 i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \log_2 i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \log_2 \frac{n}{2} = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \log_2 \frac{n}{2} \geq \frac{n}{2} \log_2 \frac{n}{2} \geq \Omega(n \log n)$$

	claim	true	false	
6.	$\frac{n}{\log n} \geq \Omega(n^{1/2})$	☒	☐	vgl. 2.
7.	$\log_7(n^8) = \Theta(\log_3(n^{\sqrt{n}}))$	☐	☒	$\lim_{n\to\infty} \frac{\log_7 n^8}{\log_3 n^{\sqrt{n}}} = \frac{8 \log_7 n}{\sqrt{n} \log_3 n} = \frac{8}{\sqrt{n}} = 0$
8.	$3n^4 + n^2 + n \geq \Omega(n^2)$	☒	☐	n^4 dominiert über n^2
9.	$n! \leq O(n^{n/2})$	☐	☒	$n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}} = n^{\frac{n}{2}} \cdot \frac{1}{2}^{\frac{n}{2}}$

nur 2. Hälfte von Produkt