Exercise 1.2 *Sum of reciprocals of roots* **(1 point).**

Consider the following claim:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq \sqrt{n}.$$

A student provides the following induction proof. Is it correct? If not, explain where the mistake is.

Base case: $n = 1$,

$$\frac{1}{\sqrt{1}} \leq 1, \text{ which is true.}$$

Induction hypothesis: Assume the claim holds for $n = k$, i.e.

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k}.$$

Induction step: Then, starting from the claim we need to prove for $n = k + 1$ and using logical equivalences:

$$\begin{aligned}\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \sqrt{k+1} &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k+1} - \frac{1}{\sqrt{k+1}} \quad \checkmark\\ &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k+1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+1}} \quad \checkmark\\ &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}} \leq \sqrt{k}, \quad \text{f}\end{aligned}$$

which is true, therefore the claim holds by the principle of mathematical induction.

The last step (marked as wrong) only holds in one direction for sure namely " $\Rightarrow$ " therefore the step cannot be traced backward

Often it is best to start with what you know even though there might be some cases where we're better off to go the other direction eg. 1.4.
Be careful whenever you go backwards in your proofs!

(d)* If $f(m)$ grows asymptotically slower than $g(m)$, then $\log(f(m))$ grows asymptotically slower than $\log(g(m))$.

Proof by counterexample: $f(m) = m$; $g(m) = m^3$.

Then $\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{m}{m^3} = \lim_{m\to\infty} \frac{1}{m^2} = 0$

But $\lim_{m\to\infty} \frac{\log(f(m))}{\log(g(m))} = \lim_{m\to\infty} \frac{\log(m)}{\log(m^3)} = \lim_{m\to\infty} \frac{\log(m)}{3\cdot\log(m)} = \frac{1}{3}$

Thus, $\log(f(m))$ and $\log(g(m))$ asymptotically grow at the same speed. $\square$

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \qquad n \geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.

a) BC: $\frac{2\cdot 1-1}{2\cdot 1} = \frac{1}{2} \leq \frac{1}{\sqrt{3\cdot 1+1}} = \frac{1}{2}$ ✓

IH: Assume the property holds for some positive integer k, that is $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3\cdot n+1}}$

IS: $k \Rightarrow k+1$

$$\underbrace{\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2k-1}{2k}\cdot\frac{2(k+1)-1}{2(k+1)}}_{a} \overset{\text{IH.}}{\leq} \frac{1}{\sqrt{3k+1}}\cdot\frac{2(k+1)-1}{2(k+1)}$$

As we want to show that $a \leq \frac{1}{\sqrt{3(k+1)+1}} = \frac{1}{\sqrt{3k+4}}$ it is now sufficient to prove that

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$$

$$\Leftrightarrow \quad \frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$

$$\Leftrightarrow \quad \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4}$$

$$\Leftrightarrow \quad \frac{4k^2+4k+1}{4k^2+8k+4} \leq \frac{3k+1}{3k+4}$$

$$\Leftrightarrow \quad (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1)$$

$$\Leftrightarrow \quad 12k^3+28k^2+19k+4 \leq 12k^3+28k^2+20k+4$$

$$\Leftrightarrow \quad 0 \leq k$$

b) We do the same as above and get:

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k}}{\sqrt{3k+3}}$$

$$\Leftrightarrow \quad 3k+3 \leq 0$$

This was just an example to show that it is sometimes easier to prove a stronger property